Waveguide containing a backward-wave slab

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We have considered theoretically the waveguide properties of a plane two-layered waveguide, whose one layer is a usual magnetodielectric (forward-wave medium), but another one is a slab of so-called *backward-wave material* (BW-material), whose both permittivity and permeability are negative. We have analyzed the properties of eigenwaves in this waveguide. In particular, it was found that there exist waves of both TE and TM polarizations, whose fields decay exponentially from the interface of the two slabs inside both layers, and their slow-wave factor tends to infinity at small frequencies. Thus, this waveguiding system supports super-slow waves with extremely short wavelengthes, as compared to the free-space wavelength and the cross section size. Other peculiarities of the spectrum are also discussed.

I. INTRODUCTION

In recent years we have witnessed increased interest in electromagnetic properties of exotic media. In particular, media with both permittivity and permeability real and negative have been studied. Plane electromagnetic waves in isotropic materials with negative parameters have the oppositely directed phase vector and the Poynting vector. By this reason, such a medium is sometimes referred as *backward-wave* (BW) medium. L.I. Mandelshtam first pointed out to unusual reflection and refraction laws at interfaces between BW and conventional media[1], which was recently observed experimentally[2,3]. V.G. Veselago performed an electrodynamical study of such a medium[4], referring to it as *"left-handed medium"* and proposed, in particular, a planar slab made of this material as a focusing lens. The concept of "perfect" lens from plane plate of BW material has been developed by[5]. Ziolkowski and Heyman[6] simulated pulse propagation through a slab of BW medium, using the FDTD method and re-considered possibilities to design the "perfect" lens.

Design of media, where the phase and energy velocities point to the opposite directions has a long history. Actually, this property exists in slow-wave structures for electronic generators with extended interaction, backward-wave tubes[7], as well as in travelling wave antennas[8]. Any relations between the phase and group velocities directions, including the opposite, can be observed in two-dimensional periodic structures[9], which in the modern literature are referred as *photonic crystals*[10].

However, all of these structures exhibit negative dispersion in such a spectral range, where the wavelength is comparable with the structure period, and it is possible to consider only their effective negative refractive index, expressing that in terms of the slow-wave factor. The realization of a composite material, where the structural sizes are much smaller than the wavelength, and an experimental verification of its properties was described in[2,3]. These new metamaterials, in a certain frequency range, can be considered as homogeneous media described by some negative permeability and permittivity parameters. A possibility for a realization of wideband composites with active inclusions was theoretically considered in[11].

N. Engheta introduced an idea to make a compact cavity resonator composed of two layers, so that one of them is a usual material, and the other one is a BW medium[12]. If this structure is inserted between two electric walls, the resonant frequencies of the cavity do not depend on the total thickness of the two-layered structure, but only on the ratio of the tangents of the thicknesses of the separate layers. Such a property suggests a possibility to realize very thin (or thick, if desired) resonators. Obviously, such a two-layered structure considered as a waveguide would exhibit some properties which are not met in waveguides composed of usual materials. In a conference presentation[13] it was pointed to some peculiarities of this waveguiding structure, in particular to the fact that in the limit of thin slabs the propagation factor approximately cancels out from the dispersion relation.

In this paper we present our results of a detailed study of wave propagation in two-layered closed waveguides whose one layer is a usual forward-wave (FW) material and the other one is a BW (or negative) material. In paper[14] it was pointed out that a passive BW medium should be dispersive and must satisfy constrains[15]

$$\frac{d[\varepsilon(\omega)\omega]}{d\omega} > 1, \qquad \frac{d[\mu(\omega)\omega]}{d\omega} > 1. \tag{1}$$

However, the spectral properties of the waveguide composed of FW-BW media were found so unusual in comparison with the case of conventional materials, that we have decided first to restrict ourselves to the simplest model of frequency-independent BW medium parameters in order to clarify the role of relations between geometrical and material parameters of the two media layers. It appears to be quite acceptable physically because a small dispersion of ε and μ is enough to satisfy inequalities (1). Also, these limitations can be overcome using metamaterials with active inclusions[11].

Probably the most important property of such a waveguide is the existence of eigenwaves whose slow-wave factor is not restricted by $\sqrt{\varepsilon\mu}$ as in usual waveguides, and whose fields decay exponentially from the media interface within both of FW and BW layers. The nature of these waves is similar to surface waves at an interface of vacuum and an isotropic plasma within the spectral range where its permittivity is negative. Negative permeability μ makes possible existence of surface wave of the other, TE polarization. However, our analysis is not restricted to super-slow waves. We show how the relations between the layer thicknesses and material parameters influence the waveguide propagation characteristics of various modes.

II. EIGENWAVES IN TWO-LAYERED WAVEGUIDE

A. General relations

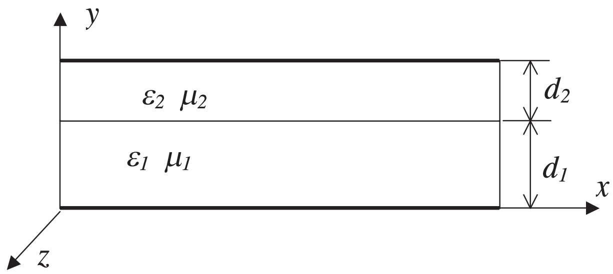

FIG. 1: Geometry of the problem: a planar waveguide filled by two slabs of different materials, one of them is a metamaterial with negative parameters.

In this paper we consider a plane two-layered waveguide, infinite along z and x directions and bounded by electric walls in $z-y$ plane at distances d_1 and d_2 from the media interface (see Fig. 1). The media are characterized by relative permittivities ε_1, ε_2 and permeabilities μ_1, μ_2. We will discuss eigenwaves propagating in z direction whose field varies depends on time and the longitudinal coordinate as $\exp(\omega t - k_z z)$.

Let us first recall the main properties of the modes propagating in the usual two-layered waveguide. Assuming that $\partial/\partial x = 0$, these waves can be separated into two classes, TE modes, whose electric field has no longitudinal and perpendicular to the interfaces components, $E_z = E_y = 0$, and TM modes with $H_z = H_y = 0$.

TE modes satisfy the dispersion relation

$$\frac{\mu_1}{k_{y1}} \tan k_{y1} d_1 + \frac{\mu_2}{k_{y2}} \tan k_{y2} d_2 = 0, \tag{2}$$

where $k_{yi} = \sqrt{k^2\varepsilon_i\mu_i - k_z^2}$ $(i = 1, 2)$, k is the wavevector in vacuum. TM modes are governed by the dispersion relation

$$\frac{k_{y1}}{\varepsilon_1} \tan k_{y1} d_1 + \frac{k_{y2}}{\varepsilon_2} \tan k_{y2} d_2 = 0. \tag{3}$$

The dominant TM_0 mode has no cutoff and its slow-wave factor has the low-frequency limit

$$n_{TM_0} \to \sqrt{\frac{d_1\mu_1 + d_2\mu_2}{d_1/\varepsilon_1 + d_2/\varepsilon_2}}, \tag{4}$$

which can be obtained by expanding the tangent functions in (3) in Taylor series or by averaging the permittivities and permeabilities within the first and second layers. Other TM and TE modes appear in pairs with the same cutoff frequencies given by relation

$$\sqrt{\frac{\mu_1}{\varepsilon_1}} \tan k\sqrt{\varepsilon_1\mu_1} d_1 + \sqrt{\frac{\mu_2}{\varepsilon_2}} \tan k\sqrt{\varepsilon_2\mu_2} d_2 = 0, \tag{5}$$

which is obtained from (2) or (3) under condition $k_z = 0$. It is important, that the dispersion properties of the waveguide modes are determined by resonance phenomena: cutoffs correspond to different standing-wave resonances

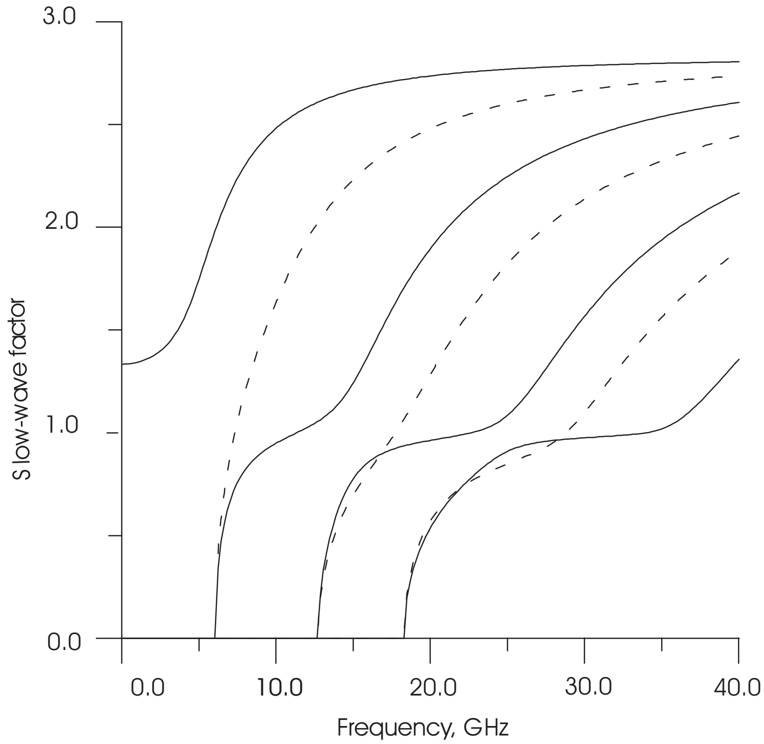

FIG. 2: Dispersion of TM modes (solid lines) and TE modes (dashed lines) in a two-layered waveguide, composed of usual (FW) materials, calculated with $\mu_1 = 2$, $\mu_2 = 1$, $\varepsilon_1 = 4$, $\varepsilon_2 = 1$, $d_1 = 0.5$ cm, $d_2 = 1$ cm.

of the cross-section $d_1 + d_2$. Let us assume, that the slow-wave factor in unbounded first-layer medium is larger than in the second layer, $\varepsilon_1\mu_1 > \varepsilon_2\mu_2$. When the frequency is only slightly over the cutoff frequency, the wave is fast, but soon its slow-wave factor becomes close to $\sqrt{\varepsilon_2\mu_2}$. The field is concentrated within the layer with the smaller value of $\varepsilon\mu$ and the field distribution is described by trigonometric functions inside both layers. Further increase of the frequency causes a re-distribution of the field, so that the slow-wave factor tends to $\sqrt{\varepsilon_1\mu_1}$, see Fig. 2. The field distribution within layer 2 is now described by exponential functions. However, equations for TM modes (3) and for TE modes (2) have wave solution only when at least one of the tangent arguments in Eqs. (2,3,5) is real, $k_{y1}^2 > 0$ in our case. It is very important in our contents that in the guide filled with conventional media the slow-wave factor never exceeds the largest value of the two refractive indices $\sqrt{\varepsilon_{1,2}\mu_{1,2}}$.

B. Backward-wave layer case: general properties

What happens, if one of the layers is a BW material? Let us assume that $\varepsilon_1 < 0$, $\mu_1 < 0$. The dispersion relations for TE and TM modes become

$$\frac{|\mu_1|}{k_{y1}} \tan k_{y1}d_1 - \frac{\mu_2}{k_{y2}} \tan k_{y2}d_2 = 0 \tag{6}$$

and

$$\frac{k_{y1}}{|\varepsilon_1|} \tan k_{y1}d_1 - \frac{k_{y2}}{\varepsilon_2} \tan k_{y2}d_2 = 0. \tag{7}$$

The same minus sign appears in the cutoff relation (5), as it follows from (6,7) at $k_z = 0$. Now real solutions of Eqs. (2,3,5) are permitted with both k_{y1}, k_{y2} being purely imaginary numbers. This means that the surface waves, whose fields decay exponentially from the interface between FW and BW layers can propagate in such a waveguide and there are no upper restrictions for their propagation constants.

In this respect, let us consider again Eq. (4), the low-frequency limit of the slow-wave factor for the fundamental

mode. If the media parameters are all positive, this value is always within the limits

$$\sqrt{\varepsilon_2\mu_2} \le \sqrt{\frac{d_1\mu_1 + d_2\mu_2}{d_1/\varepsilon_1 + d_2/\varepsilon_2}} \le \sqrt{\varepsilon_1\mu_1}. \tag{8}$$

However, if we allow negative values of the material parameters, there are no limits at all:

$$0 \le \sqrt{\frac{d_1\mu_1 + d_2\mu_2}{d_1/\varepsilon_1 + d_2/\varepsilon_2}} \le \infty. \tag{9}$$

Very peculiar situations take place in the limiting cases. If

$$d_1/\varepsilon_1 + d_2/\varepsilon_2 \to 0, \tag{10}$$

we observe that the capacitance per unit length of our transmission line (we now consider the quasi-static limit) tends to infinity. This means that although the voltage drop between the plates tends to zero, the charge density on the plates remains finite. This can be understood from a simple observation that if we fix the charge densities (positive on one plate and negative on the other), the displacement vector is fixed and, in the quasi-static limit, it is constant across the cross section. However, the electric field vector is oppositely directed in the two slabs, if one of the permittivities is negative. In the limiting case (10) the total voltage tends to zero. Similarly, in the limiting case $d_1\mu_1 + d_2\mu_2 \to 0$, the inductance per unit lengths tends to zero.

Another interesting observation concerns the case when both layer thicknesses tend to infinity, that is, the case of waves travelling along a planar interface between two media. The dispersion equations reduce to

$$\frac{|\mu_1|}{k_{y1}} - \frac{\mu_2}{k_{y2}} = 0, \quad \text{TE modes}, \qquad \frac{k_{y1}}{|\varepsilon_1|} - \frac{k_{y2}}{\varepsilon_2} = 0, \quad \text{TM modes} \tag{11}$$

It is well known (and obvious from the above relations) that surface waves at an interface can exist only if at least one of the media parameters is negative, an obvious example is an interface with a free-electron plasma region. If both parameters are negative, both TE and TM surface waves can exist. A very special situation realizes if the parameters of the two media differ only by sign, that is, if $\varepsilon_1 = -\varepsilon_2$ and $\mu_1 = -\mu_2$. In this case the propagation factor cancels out from the dispersion relations, because $k_1 = k_2$. This means that waves with any *arbitrary* value of the propagation constant are all eigenwaves of the system at the frequency where this special relation between the media parameters is realized. A similar observation was made in[13] as an approximation in case of small heights $d_{1,2}$. For a media interface this result is exact.

Next, let us study how the cutoff frequencies depend on the layer thickness. Let us fix the thickness of the first, BW layer, and consider dependence of the cutoff frequencies on the thickness of the second layer. The results are shown in Fig. 3. The cutoff frequencies F_c for the usual two-layer waveguide are presented also for comparison (dashed curves). When $d_2 = 0$, the cutoff frequencies are equal for the waveguides filled with BW and FW materials. An increase of d_2 causes a decrease of the effective thickness of the two-layered waveguide if the first layer is a BW medium. This leads to an increase of F_c. In other words, we can say that the cutoff corresponds to a resonance condition for waves travelling in the vertical direction, along y axis. Since the phase velocity is directed oppositely in the two layers, the phase shift is partially compensated, hence the electric thickness gets smaller and the cutoff frequency increases (see solid curves in Fig. 3). However, further increase of d_2 leads to a compensation of this negative negative contribution to the phase shift, and F_c becomes again smaller.

III. TE MODES

Let us consider dispersion of TE modes (see Fig. 4), calculated with different thicknesses of the second (FW) layer. One notable difference from the usual two-layered waveguide is a change of dispersion sign (see curves 1,2). It is caused by the opposite directions of the longitudinal components of the energy transport within FW and BW layers. The frequency point where dispersion changes sign, corresponds to the situation when the total energy flows are equal within the first and second layers. The upper parts of the dispersion characteristics are nearly the same for waveguides with different d_2, because in this case the field is concentrated mainly within the first layer. However, the lower parts depend on d_2, see solid, dotted and dashed lines in Fig. 4.

Another important new feature, already noted above, is a possibility of propagation of waves whose slow-wave factor exceeds $\sqrt{\varepsilon_1\mu_1}$ (we assume, as before, that $\varepsilon_1\mu_1 > \varepsilon_2\mu_2$). Dispersion characteristics of such a super-slow wave

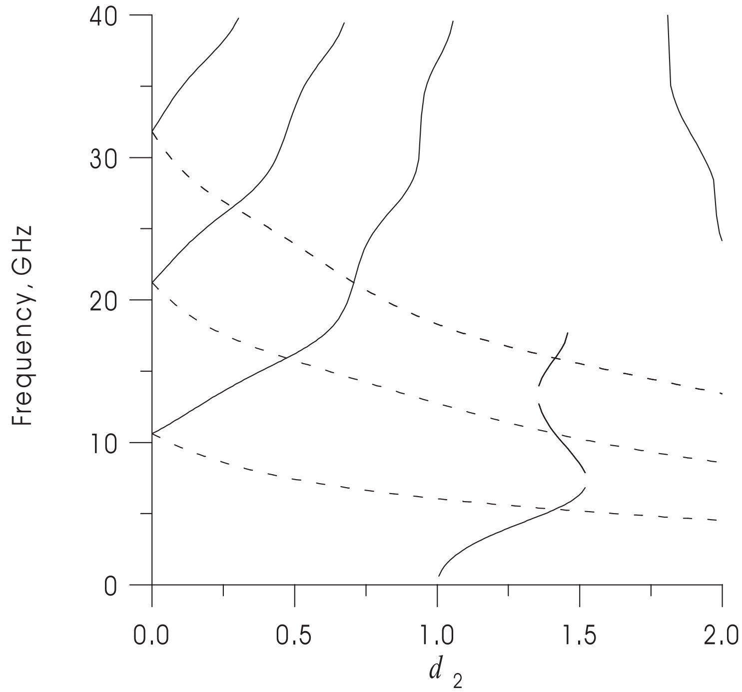

FIG. 3: Cutoff frequencies versus d_2 calculated for fixed $d_1 = 0.5$ cm, when the first layer is a FW medium (dashed line) and BW medium (solid line). The material parameters of the layers are $\mu_1 = \pm 2$; $\mu_2 = 1$; $\varepsilon_1 = \pm 4$; $\varepsilon_2 = 1$.

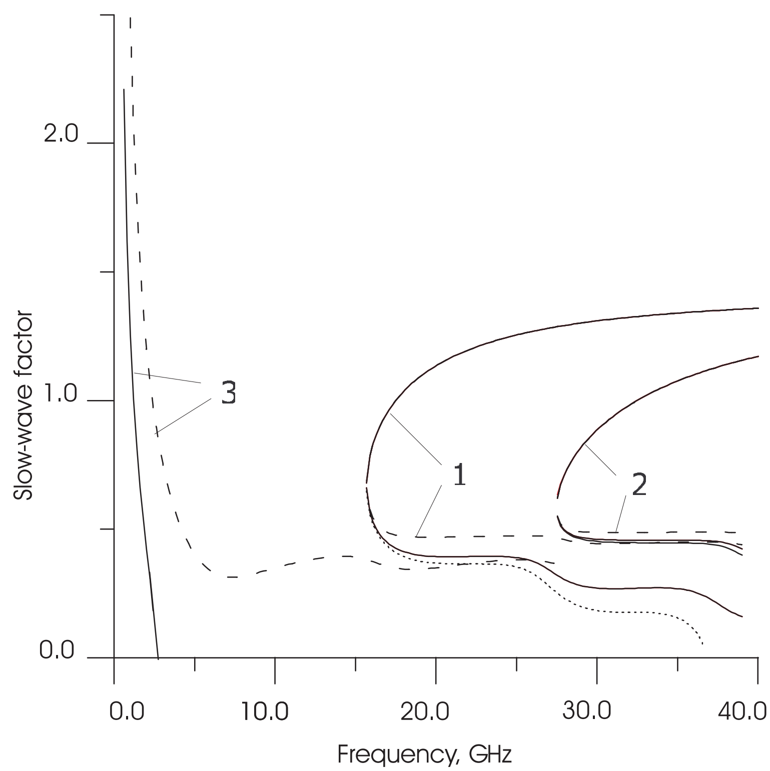

FIG. 4: Dispersion of TE modes, when the first layer is a BW material and the second layer is a usual material: $\mu_1 = -2$; $\mu_2 = 1$; $\varepsilon_1 = -4$; $\varepsilon_2 = 1$, $d_1 = 0.5$ cm. The thicknesses of the second layer are: $d_2 = 1$ cm (dotted line), $d_2 = 1.1$ cm (solid line), $d_2 = 2$ cm (dashed line).

for different d_2 are shown by curves 3, Fig. 4. We observe that the slow-wave factor tends to infinity if $k \to 0$. When its propagation is possible? Assuming $k \to 0$, $k_z \neq 0$, Eq. (6) becomes

$$F(k_z) = \frac{|\mu_1| \tanh k_z d_1 - \mu_2 \tanh k_z d_2}{k_z} = 0. \tag{12}$$

Consider two limiting cases, when $k_z d_{1,2}$ are small or large. In the limit $k_z d_{\min} \to \infty$ ($d_{\min} = \min(d_1, d_2)$), $F(k_z) > 0$ due to the assumption $|\mu_1| > \mu_2$. For the second limiting case there may be two possibilities, when $k_z = 0,\ k \neq 0$ and $k_z/k \to n_0 \neq 0,\ k \to 0$. If $k_z = 0,\ k \neq 0$ we can expand the tangents in Taylor series and obtain $\tan k_{y1} d_1 \simeq k\sqrt{\varepsilon_1 \mu_1} d_1$, $\tan k_{y2} d_2 \simeq k\sqrt{\varepsilon_2 \mu_2} d_2$, $F(k_z) \approx |\mu_1| d_1 - \mu_2 d_2$. Obviously, $F(k_z)$ can be negative for small k_z only if the condition

$$|\mu_1| d_1 < \mu_2 d_2, \qquad (\text{for} \quad |\mu_1| > \mu_2) \tag{13}$$

is satisfied. In this case function $F(k_z)$ changes sign and there is a solution of (6). Next let k_z/k be equal to some nonzero value n_0 at $k \to 0$. Then $k_{y1} = k\sqrt{\varepsilon_1 \mu_1 - n_0^2} \to 0$, $k_{y2} = k\sqrt{\varepsilon_2 \mu_2 - n_0^2} \to 0$, and we come again to relation (13).

Numerical solution of equation $F(k_z) = 0$ confirms existence of nonzero roots k_z, corresponding to very large slow-wave factors if condition (13) is satisfied. In our example, just when $\mu_2 d_2$ becomes larger than $\mu_1 d_1 = 1$ cm (we have taken $\mu_2 d_2 = 1.1$ cm), a new mode appears (see curve 3, solid line). Further increase of the thickness d_2 causes a shift of the dispersion curve to higher frequencies (dashed curve).

Still another peculiarity of the spectrum of two-layered waveguides filled with FW-BW materials is the existence of a non-dispersive wave (recall our assumption that both of FW and BW materials are non-dispersive). To study this possibility, let us first note that non-dispersive solutions are only possible if the arguments of the two tangent functions in (6) are equal (so that there is no dependence on the wavenumber k). If this condition is satisfied, the tangent functions can be cancelled, and the eigenvalue equation (6) can be easily solved. The result for the slow-wave factor reads

$$n_c = \sqrt{\frac{|\mu_1|\mu_2(|\mu_1|\varepsilon_2 - \mu_2|\varepsilon_1|)}{\mu_1^2 - \mu_2^2}}. \tag{14}$$

Next, let us check if the arguments of the tangent functions can be indeed equal for this solution. Substituting (14), the arguments of the tangent functions in (6) read

$$\sqrt{\frac{\varepsilon_1\mu_1 - \varepsilon_2\mu_2}{\mu_1^2 - \mu_2^2}}|\mu_1| d_1 \quad \text{and} \quad \sqrt{\frac{\varepsilon_1\mu_1 - \varepsilon_2\mu_2}{\mu_1^2 - \mu_2^2}}\mu_2 d_2. \tag{15}$$

These values must be equal for a non-dispersive solution. We conclude that the non-dispersive wave with the propagation factor given by (14) exists if the following two conditions are satisfied:

$$\begin{aligned} &|\mu_1| d_1 = \mu_2 d_2 \\ &(|\mu_1|\varepsilon_2 - \mu_2|\varepsilon_1|)(\mu_1^2 - \mu_2^2) > 0. \end{aligned} \tag{16}$$

The first condition guaranties the existence of a solution, and if the second condition is satisfied, the solution is a real number corresponding to a propagating mode. It can be seen also that such a solution describes a surface wave with an exponential field distribution in both of the media if

$$(\mu_1^2 - \mu_2^2)(\varepsilon_1\mu_1 - \varepsilon_1\mu_1) < 0. \tag{17}$$

In other cases the field distribution is described by trigonometric functions.

Let us discuss the properties of this non-dispersive wave. Its propagation constant does not depend on the frequency, which means that such a wave has no cutoff. Furthermore, its existence is not connected with the total thickness $d_1 + d_2$, but only with the relation $d_2/d_1 = |\mu_1|/\mu_2$. It is illustrated by curve 1, Fig. 5, calculated at $d_1 = 0.1$ cm. No other waves propagate within the spectral range presented in Fig. 5. This wave disappears under any small deviation of either d_2 or d_2 violating relation $|\mu_1| d_1 = \mu_2 d_2$, but it is not so sensitive to the values of ε_1 and ε_2, it is enough that inequality (17) is satisfied.

Consider next what happens under a small deviation of the second layer thickness, violating relation $|\mu_1| d_1 = \mu_2 d_2$. If, as above, $d_1 = 0.1$ cm, $d_2 = 0.2$ cm, taking d_2 smaller than 0.2 cm we find that the non-dispersive wave disappears, and no other waves propagate in the considered spectral range. Taking d_2 larger than 0.2 cm, the non-dispersive wave disappears also, but a new super-slow wave appears, whose slow-wave factor is larger than presented in Fig. 5. Let us

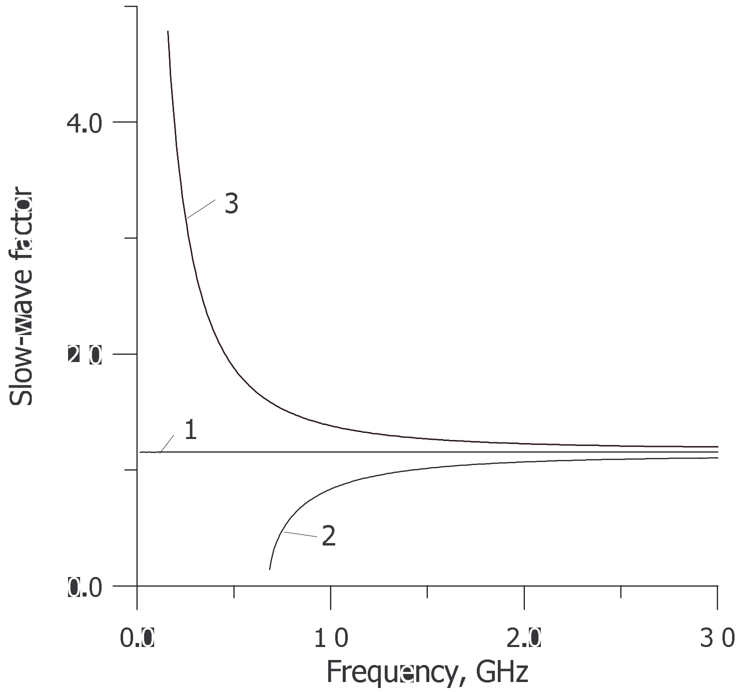

FIG. 5: Dispersion characteristics of TE modes, calculated as $\mu_1 = -2$; $\mu_2 = 1$; $\varepsilon_1 = -4$; $\varepsilon_2 = 3$, $d_1 = 0.1$ cm, d_2=0.2 cm or $d_1 = 1$ cm, d_2=2 cm (curve 1), $d_1 = 1$ cm, d_2=1.95 cm (curve 2) and $d_1 = 1$ cm, $d_2 = 2.05$ cm (curve 3).

consider thicker layers with d_1=1 cm, d_2=2 cm. Obviously, the slow-wave factor of the non-dispersive wave remains the same. Let the thicknesses of the first layers be 1 cm and that of the second be 1.95 cm. In this case the wave has a low-frequency cutoff, determined by Eq. (5), and its dispersion characteristic tends to the dispersion curve of the non-dispersive wave (see curve 2, Fig. 5). Taking d_2=2.05 cm, we observe a super-slow wave at low frequencies, whose dispersion also tends to the case of the non-dispersive wave at large frequencies (see curve 3). Thus, small deviations of d_2 dramatically change the dispersion properties of the eigenwaves at low frequencies.

IV. TM MODES

Dispersion characteristics of TM modes are presented in Fig. 6. As for TE modes, there exist modes whose slow-wave factor is restricted by $n_{\max} = \max\left(\sqrt{\varepsilon_1\mu_1}, \sqrt{\varepsilon_2\mu_2}\right)$. They may change the sign of dispersion (solid curves, Fig. 6) or not (dashed and dotted lines) depending of the value of d_2 (we assume that $d_1 = 0.5$ cm is fixed). The change of the dispersion sign is caused by the opposite directions of the energy transport within the FW and BW layers, as was discussed above for the TE modes.

Analysis of Eq. (7) shows, that, similarly to the case of TE modes, a non-dispersive TM mode exists under conditions

$$\begin{aligned} &|\varepsilon_1|d_1 = \varepsilon_2 d_2, \\ &(|\varepsilon_1|\mu_2 - \varepsilon_2|\mu_1|)(\varepsilon_1^2 - \varepsilon_2^2) > 0. \end{aligned} \tag{18}$$

Its slow-wave factor is constant and equals to

$$n_c = \sqrt{\frac{|\varepsilon_1|\varepsilon_2(|\varepsilon_1|\mu_2 - \varepsilon_2|\mu_1|)}{\varepsilon_1^2 - \varepsilon_2^2}}. \tag{19}$$

This wave is localized near the interface of the two media slabs if

$$(|\varepsilon_1| - \varepsilon_2)(|\varepsilon_1|\mu_1 - \varepsilon_2\mu_2) < 0. \tag{20}$$

In other cases the field distribution is described by trigonometric functions. As a non-dispersive TE mode, such a TM mode can exist even if the thicknesses of the layers are much smaller than the wavelength. Its dispersion is shown by curve 1, calculated for the same ε and μ as the other curves in Fig. 6.

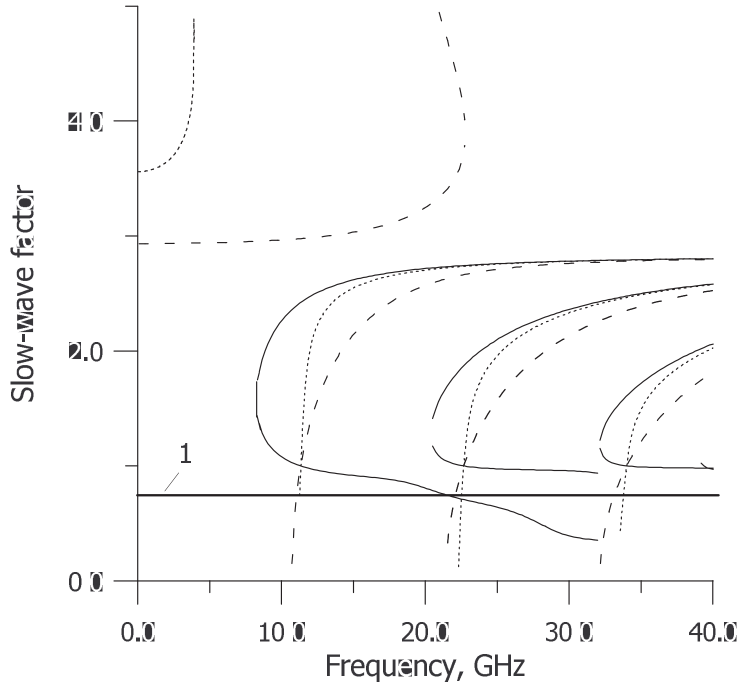

FIG. 6: Dispersion of TM modes, when the first layer is a BW material and the second layer is a usual material. The thicknesses of the second layer are: $d_2 = 1$ cm (solid line), $d_2 = 0.01$ cm (dashed line), $d_2 = 0.05$ cm (dotted line). The other parameters of the waveguide are the same as in Fig. 4. The thick curve 1 shows the non-dispersive TM mode.

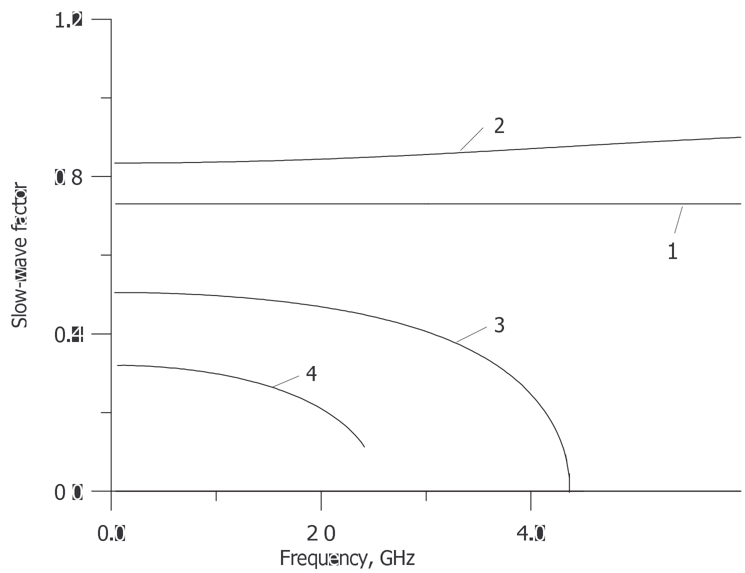

FIG. 7: Dispersion of the dominant TM_0 mode, calculated at the same $\varepsilon_{1,2}$, $\mu_{1,2}$, and d_1 as in Fig. 6 and $d_2 = 2$ cm (curve 1), $d_2 = 3$ cm (curve 2), d_2=1.3 (curve 3), $d_2 = 1.1$ cm (curve 4).

Next we consider the properties of the dominant TM_0 mode, if one of the layers is a BW material. The quasi-static limit (4) becomes

$$n_{TM_0} \to \sqrt{\frac{d_1|\mu_1| - d_2\mu_2}{d_1/|\varepsilon_1| - d_2/\varepsilon_2}}. \tag{21}$$

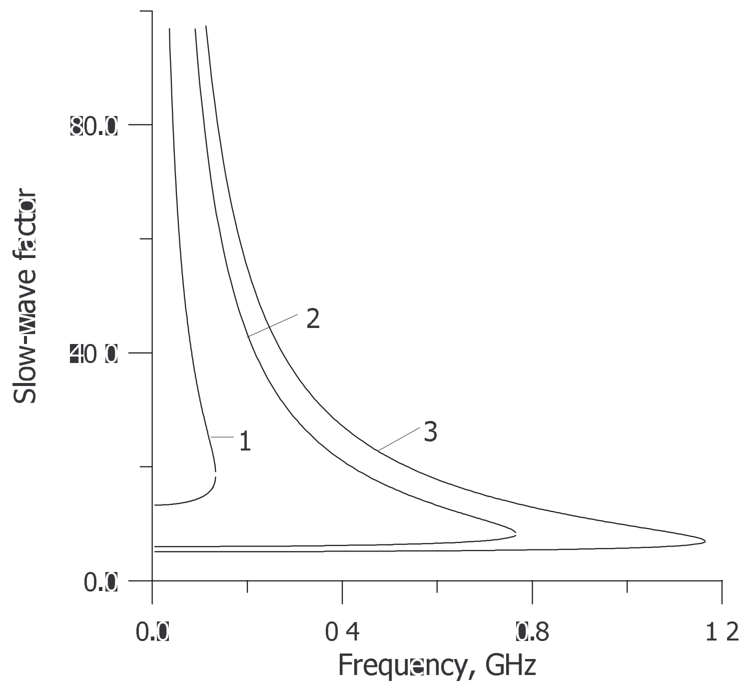

FIG. 8: Dispersion of the super-slow TM mode, calculated at $d_2 = 0.12$ cm (curve 1), $d_2 = 0.1$ cm (curve 2), and $d_2 = 0.09$ cm (curve 3).

However, now such a wave can propagate only under condition

$$(d_1|\mu_1| - d_2\mu_2)(d_1/|\varepsilon_1| - d_2/\varepsilon_2) > 0, \tag{22}$$

when the square root is real. As already noted above, in contrast to the ordinary waveguide, where

$$\sqrt{\varepsilon_2\mu_2} < n_{TM_0} < \sqrt{\varepsilon_1\mu_1} \tag{23}$$

(under assumption $\sqrt{\varepsilon_2\mu_2} < \sqrt{\varepsilon_1\mu_1}$), the value n_{TM_0} given by (22) is not restricted by relation (23). Let us fix d_1 and consider how the dispersion of this wave changes in dependence of d_2. If $d_2 > d_1|\varepsilon_1|/\varepsilon_2 = 2$ cm, the slow-wave factor increases with d_2 and tends to 1 if $d_2 \to \infty$ at small frequencies. It is illustrated by Fig. 7, where curve 1, calculated at d_2=2 cm, corresponds to the non-dispersive wave, and curve 2 corresponds to a larger value of the second layer thickness d_2=3 cm.

The slow-wave factor decreases with d_2 while condition (22) is satisfied, that is $d_2 > d_1|\mu_1|/\mu_2 = 1$ cm, but $d_2 < d_1|\varepsilon_1|/\varepsilon_2 = 2$ cm for chosen parameters. In this case a high-frequency cutoff is observed, see curves 3,4 in Fig. 7.

Thus, an important difference between non-dispersive TM and TE modes is that the TE mode disappears at small frequencies if the relation $|\mu_1|d_1 = \mu_2 d_2$ is violated (see Fig. 5). In contrast, the non-dispersive TM mode under violation of the condition $|\varepsilon_1|d_2 = \varepsilon_2 d_1$ continuously changes its dispersion and exists at $k \to 0$.

Another possibility of the existence of a wave without a low-frequency cutoff, following from (22), is the condition $d_2 < d_1\varepsilon_2/|\varepsilon_1|$. This case is especially important, because there propagation of a super-slow mode is permitted, like for the TE mode. Let us show, that existence of such a wave becomes indeed possible. Consider the function which determines the left-hand side of Eq. (7):

$$F(k_z) = \frac{k_{y1}}{|\varepsilon_1|}\tan k_{y1}d_1 - \frac{k_{y2}}{\varepsilon_2}\tan k_{y2}d_2. \tag{24}$$

In the limiting case $k_z d_{\min} \gg 1$ ($d_{\min} = \min(d_1, d_2)$), $F(k_z) < 0$ because $|\varepsilon_1| > \varepsilon_2$. Since there is a solution $k_z = kn_{TM_0}$ at $k \to 0$, we can expand the tangents in Taylor series at small k: $\tan x \sim x + x^3/3 + ...$ Taking into account that

$$\begin{aligned} \varepsilon_1\mu_1 - n_{TM_0} &= \tfrac{|\varepsilon_1|d_2(\varepsilon_1\mu_1 - \varepsilon_2\mu_2)}{d_2|\varepsilon_1| - d_1\varepsilon_2}, \\ \varepsilon_2\mu_2 - n_{TM_0} &= \tfrac{\varepsilon_2 d_1(\varepsilon_1\mu_1 - \varepsilon_2\mu_2)}{d_2|\varepsilon_1| - d_1\varepsilon_2}, \end{aligned} \tag{25}$$

and $\tan k_{y1}d_1 \approx k\sqrt{\varepsilon_1\mu_1 - (n_{TM_0})^2}d_1$, $\tan k_{y2}d_2 \approx k\sqrt{\varepsilon_2\mu_2 - (n_{TM_0})^2}d_2$, we find that the first term of the expansion gives zero value of $F(k_z)$, and taking the second term we obtain

$$F(k_z) \approx \frac{1}{3}\frac{|\varepsilon_1|\mu_1 - \varepsilon_2\mu_2}{d_2|\varepsilon_1| - d_1\varepsilon_2} d_1^2 d_2^2 \left(|\varepsilon_1|d_1 - \varepsilon_2 d_2\right). \tag{26}$$

Since we have assumed $|\varepsilon_1| > \varepsilon_2$, the condition $F(k_z) > 0$ at small $k_z d_{\min}$ becomes

$$(|\varepsilon_1|d_2 - \varepsilon_2 d_1)(|\varepsilon_1|d_1 - \varepsilon_2 d_2) > 0. \tag{27}$$

This can be realized if both expressions in the brackets are of the same sign. Numerical calculations confirm existence of a wave, whose slow-wave factor dramatically increases with $k \to 0$ only when

$$\begin{aligned} |\varepsilon_1|d_2 - \varepsilon_2 d_1 > 0, \\ |\varepsilon_1|d_1 - \varepsilon_2 d_2 > 0, \end{aligned} \tag{28}$$

which in our case corresponds to $d_2 < 0.125$. The super-slow wave is seen at Fig. 6, dashed and dotted curves. Fig. 8 illustrates dispersion properties of super-slow TM-modes, calculated for different d_2 at small frequencies. The smaller is d_2, the larger frequency band where such a mode is observed. Numerical analysis has shown the absence of super-slow solutions for the other pair of conditions, giving $F(k_z) > 0$, namely,

$$\begin{aligned} |\varepsilon_1|d_2 - \varepsilon_2 d_1 < 0, \\ |\varepsilon_1|d_1 - \varepsilon_2 d_2 < 0. \end{aligned} \tag{29}$$

In this case wave solutions also exist, but they are normal waves with moderate propagation constants and weak dispersion (at low frequencies).

V. CONCLUSION

We have considered the eigenmodes in a layered waveguide containing a layer of a backward-wave metamaterial, which has negative and real material parameters. We have found important differences between the eigenmode spectra in ordinary and FW-BW two-layered waveguide, which can be summarized as following:

1. In a FW-BW waveguide both TE and TM modes can change the dispersion sign. This is possible because the energy transport directions are opposite in FW and BW layers, so there exists a spectral point, where the power flows in the two layers compensate each other.

2. Under certain relations between the permeabilities and thicknesses of FW and BW layers there exists a non-dispersive TE mode without a low-frequency cutoff. Its slow-wave factor is constant and does not depend on the layers thicknesses.

3. In contrast with the ordinary two-layered waveguide, where always exists a dominant TM_0 mode without a low-frequency cutoff, in the FW-BW waveguide its analog disappears under certain conditions. In addition, such a wave can be non-dispersive under certain relations between the permittivities and thicknesses of FW and BW layers. Furthermore, this mode has a high-frequency cutoff under certain conditions.

4. There exist both TE and TM super-slow waves, whose slow-wave factor is not restricted by the values of the permittivities and permeabilities of the layers. The fields of these waves decay exponentially in both FW and BW layers from their interface in case of large propagation constants. It is remarkable, that such super-slow modes are caused not by large values of the permeability or permittivity, like it takes place near a resonance in ferrite or plasma, but by the layer thickness effects.

   The super-slow TE mode has a high-frequency cutoff at k_z=0, or its slow-wave factor tends to a limit, determined by the non-dispersive TE mode.

   The slow-wave factor of the super-slow TM mode has a bottom restriction, determined by the dominant TM_0 mode.

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